values of u at points in $\mathcal{R}$ in a network of squares with spacing $h = 1/3$.

5. Solve the Laplace equation $\nabla^2 u = 0$. The boundary conditions are $u = 0$ on the inner boundary and $u = 1$ on the outer boundary (see Fig. 7.14)
   (i) Give the linear equation system for the u-values at the nodes when the 5-point scheme is used. (BIT 18 (1978), 366)
   (ii) Write the Gauss-Seidel iteration procedure for solution of the system of equations.

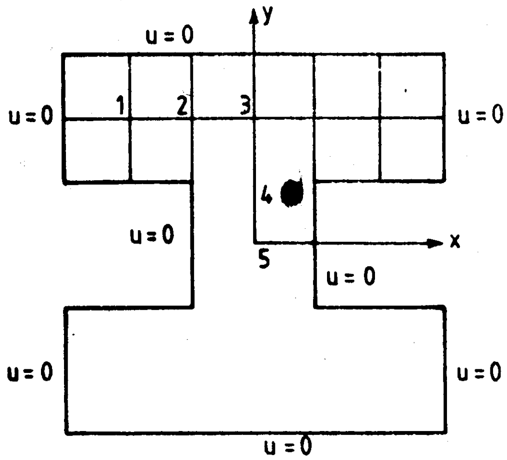

Fig. 7.11 Mesh points

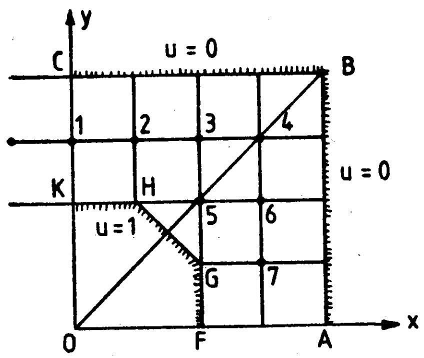

Fig. 7.12 Mesh points

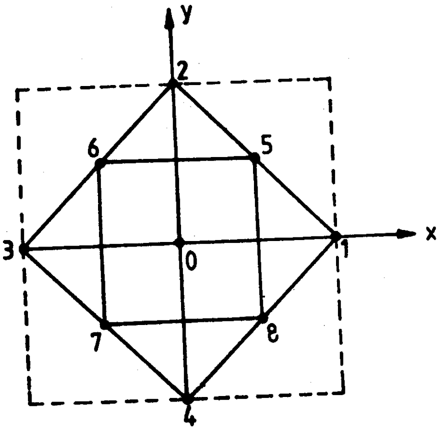

Fig. 7.15 Mesh points

9. Use the network consisting of the equilateral triangles with side h as shown in the Figure 7.16, to obtain the following difference approximations:

(i) $h^2\nabla^2 u_0 = \frac{2}{3}(\sum_{i=1}^{6} u_i - 6u_0)$

(ii) $h^4\nabla^4 u_0 = \frac{16}{9}(12u_0 - 3\sum_{i=1}^{6} u_i + \sum_{j=7}^{12} u_j)$

(iii) Find the solution of the boundary value problem $\nabla^2 u = -1$ on an equilateral triangle of side 4 with $u = 0$ on the boundary with $h = 1$.

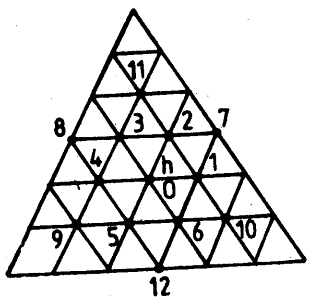

Fig. 7.16 Equilateral triangular network

10. We cover the plane with regular hexagons as shown in Fig. 7.17 with side h, and we take the vertices to be the mesh points of a hexagonal grid. Prove that the difference approximation for the Laplace operator is given by

$$h^2\nabla^2 u_0 = \frac{4}{3}(\sum_{i=1}^{3} u_i - 3u_0)$$

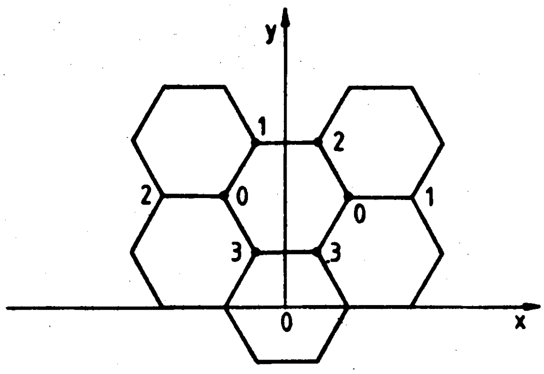

Fig. 7.17 Hexagonal network

11. In a general triangular network with coordinates and dimensions as shown in Figure 7.18, the operator ∇^2 can be transformed into

$$\frac{1}{2\sin\alpha_1\sin\alpha_2\sin\alpha_3}\left[\sin 2\alpha_1\frac{\partial^2}{\partial u^2}+\sin 2\alpha_2\frac{\partial^2}{\partial v^2}+\sin 2\alpha_3\frac{\partial^2}{\partial w^2}\right]$$

Using this, or otherwise, obtain a suitable difference representation for $\nabla^2\phi$ at a typical point P_0 of this network in terms of the values at P_0 and the six surrounding points. This representation of ∇^2 is used for the solution of $\nabla^2\phi = c$ inside a closed region whose boundary is made up of mesh line, with ϕ specified on the boundary. Show that the Jacobi method will converge if $0 < \alpha_1, \alpha_2, \alpha_3 < \pi/2$.

12. Obtain the solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-2\frac{\partial u}{\partial x} = -2 \text{ in } \mathcal{R}$$

$$u = 0 \text{ on } \partial\mathcal{R}$$

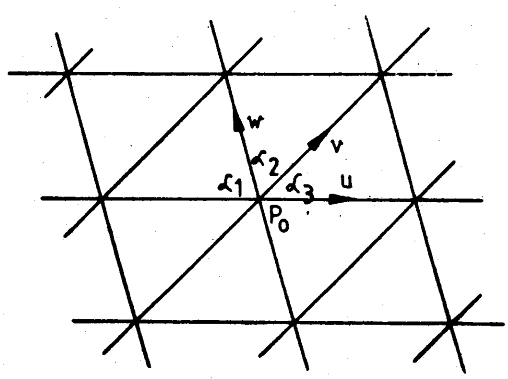

Fig. 7.18 General triangular network

is used to solve the system of linear equations. Show that for the convergence of the iterative method the optimal value of α is the smaller root of the quadratic equation

$$\left(\cos\frac{\pi}{L}+\cos\frac{\pi}{M}\right)^2\alpha^2-4\alpha+1=0$$

where L and M are the mesh divisions on the sides of the rectangle. The value of α will be between $\frac{1}{4}$ and $\frac{1}{2}$.

19. Find the solution of the differential equation

$$\nabla^2 u=-1 \qquad 0\leqslant x\leqslant 1,\quad 0\leqslant y\leqslant 1$$
$$\frac{\partial u}{\partial n}=0 \qquad \text{on} \quad x=0 \text{ and } y=0$$
$$\frac{\partial u}{\partial n}=2 \qquad \text{on } x=1$$
$$\frac{\partial u}{\partial n}=-2 \qquad \text{on } y=1$$

where n is an inward normal.
Use the 5-point difference scheme with $h=\frac{1}{2}$.

20. Solve the differential equation

$$-\nabla^2 u+0.1u=1,\quad 0\leqslant x, y\leqslant 1$$

subject to the boundary conditions

$$u=0 \text{ on } x=0 \text{ and } y=0$$
$$\frac{\partial u}{\partial n}=0 \text{ on } x=1 \text{ and } y=1$$

where n is the inward normal.
Use th 5-point difference scheme with $h=\frac{1}{2}$.

21. Solve the boundary value problem

$$\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial y^2}=-1,\quad 0\leqslant x, y\leqslant 1$$

$u=0$ on $x=0$, $y=0$, $y=1$ and $\dfrac{\partial u}{\partial n}=u$

on $x=1$, where n is the inward normal.
Use the difference scheme of $O(h^2)$ with $h=\frac{1}{2}$.

22. The function $u(x, y)$ satisfies the equation

$$-\nabla^2 u+\sigma(x, y)u=f(x, y)$$

throughout the triangular region OAB (see Fig. 7.20) together with the boundary conditions

$$u=\phi(x, y) \text{ on } OA, OB$$
$$\frac{\partial u}{\partial n}+\alpha(x, y)u=\gamma(x, y) \text{ on } AB$$

$\sigma, f, \phi, \alpha, \gamma$ being given functions, and n the outward normal.

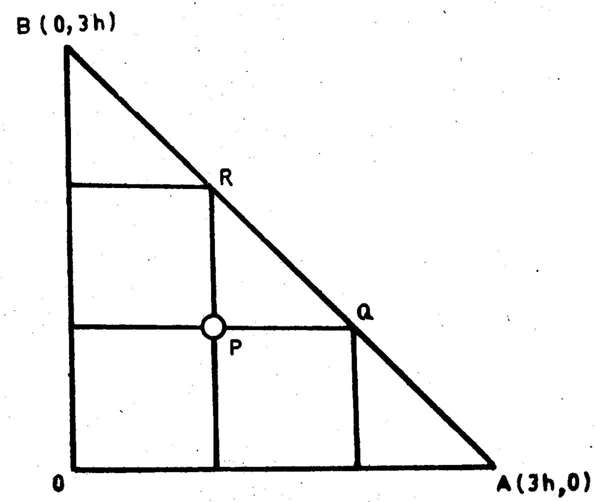

Fig. 7.20 Mesh points

By integrating over suitable regions, show that approximations to the values of u at P, Q, R are given by

$$(4+h^2\sigma_P)u_P - u_Q - u_R = h^2 f_P + \phi(h, 0) + \phi(0, h)$$
$$-u_P + (2+\sqrt{2}h\alpha_Q + \tfrac{1}{2}h^2\sigma_Q)u_Q$$
$$= \tfrac{1}{2}h^2 f_Q + \phi(2h, 0) + \sqrt{2}h\gamma_Q$$
$$-u_P + (2+\sqrt{2}\alpha_R + \tfrac{1}{2}h^2\sigma_R)u_R$$
$$= \tfrac{1}{2}h^2 f_R + \phi(0, 2h) + \sqrt{2}h\gamma_R$$

Show further that the errors in these equations are respectively of $0(h^4)$, $0(h^3)$, $0(h^3)$.

Obtain u_P in the case $\sigma = 2$, $f = -4+2(x+y)^2$,

$$\phi = x^2 \text{ on } y=0, \quad \phi = y^2 \text{ on } x=0$$
$$\alpha = \sqrt{2}, \quad \gamma = 15/\sqrt{2} \text{ and } h=1$$

23. Show that when the 5-point difference approximation is applied to the problem

$$\nabla^2 u + \Lambda u = 0 \quad \text{in } \mathcal{R}$$
$$u = 0 \quad \text{on } \partial\mathcal{R}$$

where $\mathcal{R}$ is a unit square, the characteristic values of λ correspond to the problem

$$(\delta_x^2 + \delta_y^2 + \lambda)u_{l,m} = 0, \; 1 \leqslant l, m \leqslant N$$

with $$u_{l,m} = 0, \; 0 \leqslant l, m \leqslant N+1$$

where $\lambda = h^2\Lambda$ and $h = \dfrac{1}{(N+1)}$

the corresponding system of difference equations, with step size $h = 1$.
(BIT 5 (1965), 294)

30. The Extrapolated Alternating Direction Implicit (EADI) method for solving

$$\nabla^4 u = 0$$

over the open unit circle $\mathcal{R} = [(x, y); 0 < x, y < 1]$ with appropriate boundary conditions prescribed on the boundary $\partial\mathcal{R}$, can be written in the form

$$(1 + r\delta_x^4)u_{l,m}^{(n+1/2)} = ((1 + r\delta_x^4) - wrF)u_{l,m}^{(n)}$$
$$(1 + r\delta_y^4)u_{l,m}^{(n+1)} = u_{l,m}^{(n+1/2)} + r\delta_y^4 u_{l,m}^{(n)}$$

where $u_{l,m}^{(n)}$ is the nth iteration approximation to $u_{l,m}$, $u_{l,m}^{(n+1/2)}$ is an intermediate approximation to $u_{l,m}$, r is a fixed positive iteration parameter, w is an extrapolation parameter, and

$$Fu_{l,m} = (\delta_x^2 + \delta_y^2 + 2\sigma\delta_x^2\delta_y^2)^2 u_{l,m}, \quad 0 \leq \sigma \leq \frac{1}{12}$$

Show that the optimum value of the parameter is given by

$$r = \left(4 \sin^2 \frac{\pi}{N}\right)^{-1}, \quad w = 2\left[1 + (1 - 4\sigma)^2 \sin^2 \frac{\pi}{N}\right]^{-1}$$

where $h = 1/N$ is the mesh size.

8

Finite Element Methods

8.1 INTRODUCTION

The finite difference methods described in previous chapters can be considered as a direct discretization of differential equations. In finite element methods we generate difference equations by using approximate methods with the piecewise polynomial solution. The details of this formulation will be discussed, including a brief description of the weighted residual and the variational methods. We also discuss the construction of the piecewise polynomial functions in one, two and three space dimensions. Finally we study the application of the finite element methods to the solution of ordinary and partial differential equations.

8.2 WEIGHTED RESIDUAL METHODS

The weighted residual methods are the approximate methods which provide analytical procedure for obtaining solutions in the form of functions which are close in some sense to the exact solution of the boundary value problem or the initial value problem. We formulate the weighted residual methods for the boundary value problem

$$L[u] = r(\mathbf{x}), \ \mathbf{x} \in \mathcal{R} \tag{8.1}$$

$$U_\mu[u] = \gamma_\mu, \quad \mathbf{x} \in \partial\mathcal{R} \tag{8.2}$$

where $L[u]$ denotes a general differential operator involving spatial derivatives of u; $U_\mu[u]$ represents the appropriate number of boundary conditions and $\mathcal{R}$ is the domain with boundary $\partial\mathcal{R}$. The coordinate $\mathbf{x}$ is assumed as a one dimensional coordinate in the following section, although the definition of $\mathbf{x}$ may be extended and interpreted as a coordinate in multidimensional space. The solution of the boundary value problem (8.1)-(8.2) is often attempted by assuming an approximation to the solution $u(\mathbf{x})$, an expression of the form

$$u(\mathbf{x}) \approx w(\mathbf{x}, a_1, a_2, \cdots, a_N) \tag{8.3}$$

The size of one or more subdomains decreases as N is increased, with the result that the differential equation is satisfied on the average in smaller and smaller subdomains, and hopefully the residue approaches zero everywhere.

8.2.3 Galerkin method

In the Galerkin method the weighting function is chosen to be

$$W_j = \frac{\partial w(\mathbf{x}, \mathbf{a})}{\partial a_j}, \quad j = 1, 2, \ldots, N \tag{8.15}$$

where $w(\mathbf{x}, \mathbf{a})$ is the approximate solution of the problem. Equations (8.8) in the Galerkin method become

$$\int_{\mathscr{R}} \psi_j(\mathbf{x}) E(\mathbf{x}, \mathbf{a})\, d\mathbf{x} = 0, \quad j = 1, 2, \ldots, N \tag{81.6}$$

8.2.4 Moment method

In this method, we take the weighting function

$$W_j = P_j(\mathbf{x}) \tag{8.17}$$

where $P_j(\mathbf{x})$ are polynomials. Equations (8.8) become

$$\int_{\mathscr{R}} P_j(\mathbf{x}) E(\mathbf{x}, \mathbf{a})\, d\mathbf{x} = 0, \quad j = 1, 2, \ldots, N \tag{8.18}$$

The method of moments is similar to the Galerkin method except that the residual is made orthogonal to members of a system of functions which need not be the same as the approximating function. In practice, we take $W_j = \mathbf{x}^j$, and get better results if we orthogonalize them before use.

8.2.5 Collocation method

We choose N points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ in the domain $\mathscr{R}$ and define the weighting function as

$$W_j = \delta(\mathbf{x} - \mathbf{x}_j) \tag{8.19}$$

where δ represents the unit *impulse* or *Dirac delta* which vanishes everywhere except at $\mathbf{x} = \mathbf{x}_j$. The collocation equations become

$$\int_{\mathscr{R}} \delta(\mathbf{x} - \mathbf{x}_j) E[\mathbf{x}, \mathbf{a}]\, d\mathbf{x} = 0 \tag{8.20}$$

which can be written as

$$E[\mathbf{x}_j, \mathbf{a}] = 0, \quad j = 1, 2, \ldots, N \tag{8.21}$$

This criterion is thus equivalent to putting $E[\mathbf{x}, \mathbf{a}]$ equal to zero at N points in the domain $\mathscr{R}$. The distribution of the collocation points on $\mathscr{R}$ is arbitrary. However, in practice we distribute the collocation points uniformly on $\mathscr{R}$.